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Small Values of the Maximum for the Integral of Fractional Brownian Motion

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We consider the integral of fractional Brownian motion (IFBM) and its functionals ξ_T on the intervals $(0, T)$ and $(-T, T)$ of the following types: the maximum M_T, the position of the maximum, the occupation time above zero etc. We show how the asymptotics of $P(\xi_T < 1) = p_T, T \to \infty$, is related to the Hausdorff dimension of Lagrangian regular points for the inviscid Burgers equation with FBM initial velocity. We produce computational evidence in favor of a power asymptotics for p_T. The data do not reject the hypothesis that the exponent θ of the power law is related to the similarity parameter H of fractional Brownian motion as follows: $\theta = -(1-H)$ for the interval $(-T, T)$ and $\theta = -H(1-H)$ for $(0, T)$. The point 0 is special in that IFBM and its derivative both vanish there.

KEY WORDS: Fractional Brownian motion; Burgers equation; fractality; long excursions.

1. INTRODUCTION

Sinai[19] and Frisch and associates[18] initiated in 1992 the study of fractal and multifractal properties of solutions of the inviscid Burgers equation with initial velocity $u_0(x)$ specified by a self-similar random process. That last circumstance guarantees that the solution is self-similar in the large. In particular, one could be interested in finding the Hausdorff dimension of the set of regular Lagrangian points S that describe the initial locations of those fluid particles which have not collided until a fixed time t_0. The

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original model of $u_0(x)$ was fractional Brownian motion (FBM), $b_H(x)$, with similarity parameter $0 < H < 1$.

By now the Sinai–Frisch program has been carried out for special Markovian models of $u_0(x)$ alone: Sinai[19] has found the dimension S for Brownian motion case, i.e., $u_0(x) = b_{1/2}(x)$; Bertoin[3] discovered for this case that the solution $u(t = t_0, x)$ admits of an exact probabilistic description as a Levy process. One can then find a multifractal description of the solution $x \to u(t_0, x)$ (the relevant references are refs. 6–8). Additionally, Bertoin[3] found that the Hausdorff dimension of Lagrangian regular points is h, if $u_0(x)$ is a stable Lévy process of index $\alpha = h^{-1} \in (1, 2]$ with no positive jumps (see also ref. 21).

The nonmarkovian case $u_0(x) = b_H(x)$, $H \neq 1/2$ has proved extremely difficult for analysis. Handa[5] found simple arguments to derive a lower bound on the dimension of S, namely, $\dim S \geqslant H$. The exact equality $\dim S = H$ is known as a hypothesis[18, 20] since 1992. Among methods developed for analyzing the nonmarkovian case $u_0(x)$, the Sinai approach is of particular interest. For the case $u_0(x) = b_{1/2}(x)$ this method[19] relates the estimation of the dimension of S to the asymptotic behavior of the probability

$$p_T = P\{\xi(x) < 1, x \in \Delta_T\}$$

for integral Brownian motion

$$\xi(x) = \int_0^x b_{1/2}(s)\, ds \qquad \text{and} \qquad \Delta_T = (0, T), \qquad T \gg 1. \tag{1}$$

As a matter of fact (see below), one has to deal with a problem that is rather popular in physical and technical applications: find the probability of a long excursion for a random process $\eta(x)$, i.e., $P\{\eta(x) > \eta(0), 1 < x < T\}$, $T \gg 1$. A review of the problem can be found in ref. 15. Sinai has shown that the quantity $p_T \cdot T^{1/4}$ is bounded away from 0 and ∞ as $T \to \infty$ under the conditions (1). That estimate was repeatedly refined and generalized.[10–12]

We show below that the upper bound $\dim S \leqslant H$ under the conditions $u_0(x) = b_H(x)$ follows from an estimate of p_T for the integral of fractional Brownian motion (IFBM): $\xi(x) = \int_0^x b_H(s)\, ds$ when considered in the bilaterally expanding interval $\Delta_T = (-T, T)$.

The work[16] clarifies the asymptotic problem of p_T for intervals $(0, T)$ and $(-T, T)$ in the case of fractional Brownian motion: $\xi(x) = b_H(x)$. It transpires that in this case

$$\ln p_T = -(1 - H) \ln T (1 + o(1)), \qquad \Delta_T = (0, T).$$

On the other hand, when $\Delta_T = \{x: |x| < T\}$, the leading term in the log asymptotics of p_T is independent of H. More generally, suppose $b_H(x)$, $x \in R^d$ is FBM with multidimensional time; in that case

$$\ln p_T = -d \ln T(1+o(1)), \qquad \Delta_T = \{x \in R^d : |x| < T\}.$$

The last asymptotics is due to the fact that the probability density for the position of the maximum of FBM exists in the sphere $\{|x| < 1\}$. A generalization of this fact is given below.

We present theoretical and computational evidence in favor of the following conjecture for IFBM:

$$\ln p_T = \begin{cases} -(1-H) \ln T(1+o(1)), & \Delta_T = (-T, T) \\ -H(1-H) \ln T(1+o(1)), & \Delta_T = (0, T); T \gg 1. \end{cases}$$

The first of these asymptotic expressions corroborates the hypothesis $\dim S = H$, so is not unexpected, while the second is, considering that the exponent $\theta(H) = H(1-H)$ has the point of symmetry $H = 1/2$.

Because IFBM is a self-similar process, the distribution of its maximum in $\Delta = (0, 1)$ or $(-1, 1)$, $F_{\max}(x)$, is related to p_T through $p_T = F_{\max}(T^{-(1+H)})$. Importantly, our calculation was performed for a series of statistics: the maximum $M = \max_\Delta$ IFBM; the position of M in Δ, G; the occupation time $A^+ = \int_\Delta \mathbf{1}_{\xi(x)>0}\, dx$ of IFBM above zero; and the rightmost zero of IFBM in $(0, T)$, Z. The distributions of these statistics (one should use $F_{\max}(x^{1+H})$ when M is considered) have identical asymptotics as $x \to 0$, but depend on interval type: $\Delta = (0, 1)$ or $(-1, 1)$. When $\Delta = (-1, 1)$, they provide independent evidence in favor of the hypothesis $\dim S = H$.

The rest of this paper is organized as follows. Section 2 reduces the evaluation of $\dim S \leqslant H$ to the asymptotic distributions of M, G, A^+ and Z near zero. Section 3 discusses the modeling of IFBM, while Section 4 presents numerical evaluations of the distributions listed above and some theoretical arguments to support our conclusions.

2. REGULAR LAGRANGIAN POINTS AND THE NONEXCEEDANCE OF LEVEL

We now define more exactly the notions used in the Introduction. We consider the Burgers equation

$$\partial_t u + u\, \partial_x u = \nu u_{xx}, \qquad \nu \downarrow 0, \tag{2}$$

with continuous initial conditions $u(0, x) = u_0(x)$ and the velocity potential $U(x) = \int_0^x u_0(x)\, dx = o(x^2)$, $x \to \infty$. The solution at $t_0 = 1$ has the form

$u(x) = x - a(x)$, where $a(x)$ can be found from $U(x)$ as follows. Construct a convex minorant $C(x)$ for $U(x) + x^2/2$. In that case its derivative $C'(x)$ is nondecreasing and has finite limits from the left and from the right. We now complete the definition of $C'(x)$ in continuity on the right. In that case, according to Hopf (see, e.g., refs. 19 and 22), $a(x)$ is identical with the inverse function of $C'(x)$. The set of points where $C'(x)$ is increasing, i.e., the topological support of the measure $dC'(x)$ or the closure of the set $\{a(x), x \in R\}$, defines the set S of regular Langrangian points in the Burgers problem. More precisely a Lagrangian regular point is a point in S that is isolated neither to its left nor to its right in S. The dynamics of completely inelastic particles on R^1 can be related to the Burgers equation: each infinitesimal particle located at x has a mass dx and an initial momentum $dU(x)$. On colliding the particles coalesce and continue movement following the conservation laws of mass and momentum. Initial positions of those particles that have not collided until time $t_0 = 1$ make up the set of regular Lagrangian points. The initial conditions $u_0(x)$ will be considered to be fractional Brownian motion $b_H(x)$, i.e., a Gaussian process with zero mean and structural function $E\,|b_H(x) - b_H(y)|^2 = |x-y|^{2H}$ where $0 < H < 1$. In virtue of the Kolmogorov theorem the paths of $b_H(x)$ can be treated as continuous a.s. The process $b_H(x)$ is self-similar, i.e., $b_H(\Lambda x) \stackrel{d}{=} \Lambda^H b_H(x)$, where $\stackrel{d}{=}$ denotes equality of finite-dimensional distributions.

Theorem 1. 1. The set of regular Lagrangian points in the Burgers problem (2) with $u_0(x) = b_H(x)$ has a.s. dimension H, if for any $\varepsilon > 0$ and $T \to \infty$ one of the following requirements is fulfilled:

$$\text{(A)}\quad P\left\{ y(x) := \int_0^x b_H(s)\, ds < 1,\ x \in \Delta_T \right\} < T^{-(1-H)+\varepsilon},$$

$$\text{(B)}\quad P(y(x) < 0,\ x \in \Delta_T,\ |x| > 1) < T^{-(1-H)+\varepsilon},$$

$$\text{(C)}\quad P(|G(\Delta_T)| < 1) < T^{-(1-H)+\varepsilon},$$

$$\text{(D)}\quad P\left\{ \int_{\Delta_T} \mathbf{1}_{y(x)>0}\, dx < 1,\ |G(\Delta_T)| < T \right\} < T^{-(1-H)+\varepsilon},$$

where $\Delta_T = (-T, T)$, $G(\Delta_T)$ is the position of the maximum of $y(x)$ in Δ_T.

2. If one of type A–D probabilities p_T has an asymptotics of the form $\log p_T = -\theta \log T(1 + o(1))$, the probabilities of the other types have the same asymptotics. This statement also holds for $\Delta_T = (0, T)$ with the probability $P(Z_T < 1)$ in addition to $(A–D)$, where Z_T is the rightmost zero of $y(x)$ in $(0, T)$.

The proof of the theorem will be preceded by two lemmas.

Lemma 1. $\dim S \leqslant H$, if for any $\varepsilon > 0$ there exists a $\delta_0 = \delta_0(\varepsilon)$ such that one has for arbitrary $x \in R^1$:

$$P(S \cap B(x, \delta) \neq \phi) < \delta^{(1-H)-\varepsilon}, \qquad \delta < \delta_0 \tag{3}$$

where $B(x, \delta)$ is a ball of radius δ centered at x.

Proof. Let us consider the measure $\mu(dx) = dC'(x)$ with support S. Cover a finite interval Δ with intervals $B_i(\delta)$ of length δ with overlappings of length $\delta/2$. The elements $\tilde{B}_i$ in $\{B_i(\delta)\}$ for which $\mu(B_i) > 0$ will then form a cover $S \cap \Delta$. In view of (3)

$$\begin{aligned} E \sum |\tilde{B}_i(\delta)|^{H+2\varepsilon} &= E \sum |B_i(\delta)|^{H+2\varepsilon} \mathbf{1}_{\mu(B_i) > 0} \\ &< \delta^{H+2\varepsilon} \cdot 2\,|\Delta|\,\delta^{-1} \cdot \delta^{(1-H)-\varepsilon} = c\delta^{\varepsilon} \end{aligned}$$

where $|\Delta|$ is the length of Δ. By Chebyshev's inequality

$$P\left(\sum |\tilde{B}_i|^{H+2\varepsilon} > a\right) < c\delta^{\varepsilon}/a.$$

Consider a sequence δ_n such that $\sum \delta_n^{\varepsilon} < \infty$. The Borel–Cantelli lemma then yields

$$\sum |\tilde{B}_i(\delta_n)|^{H+2\varepsilon} < a, \qquad n > n(\omega).$$

Since a is arbitrary:

$$\limsup_n \sum |\tilde{B}_i(\delta_n)|^{H+2\varepsilon} = 0 \quad \text{a.s.}$$

However, in that case one has $\dim(S \cap \Delta) \leqslant H + 2\varepsilon$. Since $\varepsilon > 0$ and Δ are arbitrary, one has $\dim S \leqslant H$.

Lemma 2. The conditions of Lemma 1 are fulfilled, if

$$P\left\{\int_0^x b_H(s)\, ds < 1, |x| < T\right\} < T^{-(1-H)+\varepsilon}, \qquad \forall \varepsilon > 0.$$

as $T \to \infty$.

Proof. The process $y(x)=\int_0^x b_H(s)\,ds+x^2/2$ can be represented in the form

$$y(x)=\int_0^c (b_H(s)+s)\,ds+(b_H(c)+c)(x-c)$$
$$+\int_c^x [b_H(s)-b_H(c)+(s-c)]\,ds=L(x')+\int_0^{x'} (\tilde b_H(s)+s)\,ds,$$

where $L(x')$ is a linear function of $x'=x-c$, and $\tilde b_H(x)=b_H(c+x)-b_H(c)\stackrel{d}{=}b_H(x)$. The convex minorants of y and $\tilde y=\int_0^{x'}(\tilde b_H(s)+s)\,ds$ differ by the linear function $L(x')$. Hence the fractal properties of the measure $\mu(dx)=dC'(x)$ are invariant under translation along the x-axis (this observation is due to U. Frisch). Consequently, it is sufficient to prove (3) for $S'=S\cap(-\delta/2,\delta/2)$.

Let $\Delta=(-\delta/2,\delta/2)$ contain a point of growth x_0 for the measure $d\mu$. That means that the curve $f(x)=U(x)+x^2/2$ and its convex minorant $C(x)$ do not lie below the tangent of $f(x)$ at the point x_0, and $C(x_0)=f(x_0)$. The event $\{x_0\in\Delta\}$, to be called A here, can be written as

$$A=\left\{\exists x_0\colon |x_0|<\delta/2;\int_0^x (b_H(s)+s)\,ds\right.$$
$$\left.\geqslant\int_0^{x_0}(b_H(s)+s)\,ds+(b_H(x_0)+x_0)(x-x_0),\ \forall x\in R^1\right\}.$$

Let us modify event A to become A_1, i.e., we assume that the equality in the formulation of A is true for $|x|<1$ only. To emphasize the fact that A_1 depends on the process $b_H(x)+x=\xi(x)$, we will write $A_1=A_1[\xi]$.

One has

$$P(A)\leqslant P(A_1)=E\mathbf{1}_{A_1[b_H+\varphi]}=E\mathbf{1}_{A_1[\tilde b_H]}\pi(\tilde b_H), \tag{4}$$

where $\varphi(x)=x$ and π is the Radon–Nikodim derivative of two Gaussian measures corresponding to the processes $\tilde b_H-\varphi$ and $\tilde b_H$ in $[-1,1]$. Note that $\tilde b_H$ is an FBM process. The function φ is smooth and vanishes at zero. For this reason the above measures are mutually absolutely continuous.[17] By the Cameron–Martin relation $\ln\pi(\tilde b_H)$ is a Gaussian variable with mean $-c_H^2/2$ and variance c_H^2, where $c_H=\|\varphi\|$ and $\|\cdot\|$ is the norm in Hilbert space H_B of functions on $\Delta=[-1,1]$ with reproducing kernel $B(x,y)=Eb_H(x)\,b_H(y)$. The constant c_H is finite and can be found in explicit form as indicated by Molchan and Golosov.[17]

Applying Hölder's inequality to the right-hand side of (4), one gets

$$P(A) < P(A_1[\tilde{b}_H])^{1-\varepsilon} (E\pi^{1/\varepsilon})^\varepsilon = P(A_1[b_H])^{1-\varepsilon} c_\varepsilon, \tag{5}$$

where $c_\varepsilon = \exp(\frac{1}{2}(\varepsilon^{-1}-1) c_H^2)$.

We now evaluate $P(A_1[b_H])$. One has

$$\begin{aligned} P(A_1[b_H]) = P\Bigg\{ &\exists x_0\colon |x_0| < \delta/2; \\ &\int_0^x b_H(s)\,ds \geqslant \int_0^{x_0} b_H(s)\,ds + b_H(x_0)(x-x_0),\ |x|<1 \Bigg\} \\ = P\Bigg\{ &\exists x_0\colon |x_0|<1/2; \int_0^x b_H(s)\,ds \geqslant a(x_0)+b(x_0)\,x,\ |x|<T \Bigg\}, \end{aligned}$$

where

$$T = \delta^{-1}, \qquad |a(x_0)| = \left| \int_0^{x_0} b_H(s)\,ds - b_H(x_0)\,x_0 \right| \leqslant 2M,$$

$$|b(x_0)| = |b_H(x_0)| \leqslant M = \max_{|x|<1/2} |b_H(x)|.$$

We will use the Fernique inequality[4]

$$P(M > \bar{c}_H u) < \exp(-u^2/2) = T^{-a}, \qquad u > u_0,$$

where $u = u_T = \sqrt{2a \ln T}$, $\bar{c}_H$ being a constant; the value of a will be chosen later on. From this it follows that

$$\begin{aligned} P(A_1[b_H]) &< P\{A_1[b_H], M < \bar{c}_H u_T\} + T^{-a} \\ &< P\left\{ \int_0^x b_H(s)\,ds > -2\bar{c}_H u_T - \bar{c}_H u_T\,|x|,\ |x|<T \right\} + T^{-a} \\ &= P\left\{ \int_0^x b_H(s)\,ds < u_T \bar{c}_H (2+|x|),\ |x| \leqslant T \right\} + T^{-a} \\ &= P\left\{ \int_0^x b_H(s)\,ds < 4\lambda_T^{-1} + 2\,|x|,\ |x| < T' \right\} + T^{-a}, \end{aligned} \tag{6}$$

where $T' = T/\lambda_T$, $u_T \bar{c}_H = 2\lambda_T^H$, $\lambda_T = \mathrm{const} \cdot (\ln T)^{1/2H}$. Here we have used the fact that $b_H(x)$ is a self-similar process and modified the interval $|x| \leqslant T$ to become $|x| < T'$.

Define the function

$$\varphi_1(x) = 2x\mathbf{1}_{|x|<1} + 2\,\mathrm{sgn}(x)\,\mathbf{1}_{|x|>1} = \frac{2}{\pi i}\int [e^{ix\lambda} - 1]\,\frac{\sin\lambda}{\lambda^2}\,d\lambda. \tag{7}$$

In that case (6) can be continued to get

$$P(A_1[b_H]) \leqslant P\left\{\int_0^x (b_H(s) - \varphi_1(s))\,ds < F(x),\ |x| < T'\right\} + T^{-a}, \tag{8}$$

where

$$F(x) = \begin{cases} -x^2 + 2\,|x| + 4\lambda_T^{-1}, & |x| < 1, \\ 1 + 4\lambda_T^{-1}, & |x| > 1. \end{cases}$$

When T is large, one has $F(x) < 2$. For this reason the last estimate will merely become less precise, when F is replaced with $F(x) = 2$. The right-hand side of (8) can be evaluated by repeating the steps that have led to (4, 5). The substitution of $\tilde{b}_H$ for $b_H - \varphi_1$ combined with Hölder's inequality yield

$$P(A_1[b_H]) < P\left\{\int_0^x b_H(s)\,ds < 2,\ |x| < T'\right\}^{1-\varepsilon} c_\varepsilon^1 + T^{-a},$$

where $c_\varepsilon^1 = \exp((\varepsilon^{-1} - 1)\,b^2/2)$, $b^2 = \|\varphi_1\|_T^2 \leqslant \|\varphi_1\|_\infty^2$. Here $\|\cdot\|_T$ is the norm on H_B for the interval $(-T, T)$. The spectral representations of the kernel

$$B(t, s) = Eb_H(t)\,b_H(s) = k_H^{-1}\int (e^{it\lambda} - 1)(e^{-is\lambda} - 1)\,|\lambda|^{-1-2H}\,d\lambda$$

and φ_1 (see (7)) yield

$$\|\varphi_1\|_\infty^2 = k_H \int \left|\frac{2\sin\lambda}{\pi\lambda^2}\right|^2 |\lambda|^{1+2H}\,d\lambda < \infty,$$

where $k_H = \int |e^{i\lambda} - 1|^2\,|\lambda|^{-1-2H}\,d\lambda$.

The final result is

$$P(A) \leqslant P(A_1[b_H])^{1-\varepsilon}\,c_\varepsilon < (p_{T'}^{1-\varepsilon} c_\varepsilon^1 + T^{-a})^{1-\varepsilon}\,c_\varepsilon, \tag{9}$$

where $p_{T'} = P\{\int_0^x b_H(s)\,ds < 2,\ |x| < T'\}$.

Let $p_T < T^{-(1-H)+\varepsilon_1}$ for large T. Take $a > 1-H$ and choose ε from the requirement $c_\varepsilon^{\frac{1}{\varepsilon}} \cdot c_\varepsilon = T'^{\varepsilon_1}$, i.e., $\varepsilon = c\varepsilon_1^{-1}/\ln T'$. Inequality (9) can then be continued:

$$P(A) < c_1 T'^{-(1-H)+2\varepsilon_1},$$

where $c_1 = \exp(2(1-H)\, c\varepsilon_1^{-1})$. Recalling that $T' = T(\ln T)^{-1/2H} \cdot c_2$, one obtains the desired estimate $P(A) < T^{-(1-H)+3\varepsilon_1}$, $T > T_0(\varepsilon_1)$, $T = \delta^{-1}$.

Proof of Theorem 1. The inequality $\dim S \geqslant H$ was derived by Handa.[5] The opposite inequality $\dim S \leqslant H$ follows from Lemmas 1 and 2 and condition A of Theorem 1. To prove the theorem under condition B, we note that the event $\{\int_0^x b_H(s)\, ds < c,\, |x| < T\}$ can be represented as

$$\left\{\int_0^x (b_H(s) - \varphi(s))\, ds < \psi(x),\, |x| < T\right\},$$

where φ, ψ are smooth finite functions: $\psi \equiv 0$ when $|x| \geqslant 1$ and $\psi > 0$ when $|x| < 1$, while $\varphi = 0$ when $|x| < 1/2$ and $|x| > 1$. Repeating the translation procedure for the samples: $b_H(s) - \varphi(s) \to \tilde{b}_H(s)$ and using Hölder's inequality, we get

$$\begin{aligned}
p_T &:= P\left\{\int_0^x b_H(s)\, ds < 1,\, |x| < T\right\} \\
&< c_\varepsilon P\left\{\int_0^x b_H(s)\, ds < \psi(x),\, |x| < T\right\}^{1-\varepsilon} \\
&< c_\varepsilon P\left\{\int_0^x b_H(s)\, ds < 0,\, 1 < |x| < T\right\}^{1-\varepsilon},
\end{aligned}$$

where $c_\varepsilon = \exp(\frac{1}{2}\varepsilon^{-1} \cdot c_\varphi^2)$, $c_\varphi < k_H \int |\hat{\varphi}(\lambda)|^2\, |\lambda|^{1+2H}$ and $\hat{\varphi}$ is the Fourier transform of φ. One has $c_\varphi < \infty$, because φ is smooth and finite. Choose $\varepsilon = \varepsilon_T$ from the requirement $c_\varepsilon = \mathscr{L}_T$, where $\mathscr{L}_T$ is a slowly varying function. Take $\mathscr{L}_T = \ln T$, say, then $\varepsilon_T^{-1} = c \ln \ln T$. The result is

$$p_T < \mathscr{L}_T P\left(\int_0^x b_H(s)\, ds < 0,\, 1 < |x| < T\right)^{1-\varepsilon_T}.$$

When (B) holds, one has

$$p_T < (T^{-(1-H)+\varepsilon_1})^{1-\varepsilon_T}\, \mathscr{L}_T < T^{-(1-H)+\varepsilon_2}, \qquad T \gg 1,$$

i.e., the implication $(B) \to (A)$ is true.

The inequality

$$P\left(\int_0^x b_H(s)\,ds<0,\ 1<|x|<T\right)<P(|G(-T,T)|<1),$$

where $G(\Delta)$ is the position of the maximum of IFBM in Δ, yields the implication $(C)\to(B)$. Lastly, under (B) the position of the maximum of IFBM is $|G|<1$, while the occupation time of IFBM above 0 is below 2. Hence $(D)\to(B)$.

Let us prove $(A)\to(C)$. Below, G_T is the position of the maximum of IFBM in $(-T,T)$ and $M_a=\max_{|x|<a}$ IFBM. One has

$$P(|G_T|<1)<P(|G_T|<1,\ M_1<c_T)+P(M_1>c_T).$$

If $c_T=\sqrt{2a\ln T}$, then the Fernique estimate[4] yields

$$P(M_1>c_T)<cT^{-a'},\qquad T>T_0,$$

where $a'=a/\sigma^2$ and $\sigma^2=E\,|\mathrm{IFBM}(1)|^2=(2H+2)^{-1}$. Also,

$$P(|G_T|<1,\ M_1<c_T)<P(M_T<c_T).$$

Since IFBM is self-similar, one has $M_T\overset{\mathrm{d}}{=}\lambda^{1+H}M_{T'}$, when $T=\lambda T'$. Take λ from the requirement $\lambda^{1+H}=c_T$. Then $P(M_T<c_T)=P(M_{T'}<1)$.

To sum up,

$$P(|G_T|<1)<P(M_{T'}<1)+o(T^{-a}),$$

where $T'=cT(\ln T)^{-\rho}$, $\rho=0.5(1+H)^{-1}$, while the parameter $a>0$ is arbitrary. When $a>(1-H)$, the implication (A) $\to$ (C) is obvious.

We are going to prove (A) $\to$ (D). Let A_T^+ be the occupation time of $y(x)=\mathrm{IFBM}(x)$ above zero in $\Delta_T=(-T,T)$. One has

$$P(A_T^+<1,\ |G_T|<T)\leqslant P(M_T<c_T)+P(M_T>c_T,\ A_T^+<1,\ |G_T|<T),$$

where c_T will be specified below.

Let $\Delta_T=\bigcup\Delta_k$, $\Delta_k=(k,k+1)$, and $M_k=\max\{y(x),\,x\in\Delta_k\}$. If the event $\mathscr{B}=\{M_T>c_T,\ A_T^+<1,\ |G_T|<T\}$ occurs, one will have the following for the interval Δ_k which contains G_T: $M_k>c_T$, $y(x)$ and $y'(x)=b_H(x)$ have zeroes in Δ_k. Indeed, if $y(x)\neq 0$, then $y(x)>0$ in Δ_k and $A_T^+\geqslant 1$. Consequently,

$$P(\mathscr{B})<\sum_k P\{\max((y(x_1)-y(x_2))>c_T,\ x_1,x_2\in\Delta_k),\ S_k\}:=\sum_k p_k,$$

where S_k means that $b_H(x)$ has a zero in Δ_k.

We are going to evaluate p_k:

$$p_k < P\{\max[(y(x_1) - y(x_2)), x_1, x_2 \in \Delta_k] > c_T, |b_H(k)| < c_T/2\}$$
$$+ P\{|b_H(k)| > c_T/2, S_k\} := p_{k,1} + p_{k,2}.$$

One has

$$p_{k,2} < P\{\max(|b_H(x_1) - b_H(x_2)|, x_1, x_2 \in \Delta_k) > c_T/2\}$$
$$= P\{\max(|b_H(x_1) - b_H(x_2)|, x_1, x_2 \in \Delta_0) > c_T/2\}.$$

Here we have used the fact that $b_H(x)$ has stationary increments. In virtue of the Fernique inequality[(4)]

$$p_{k,2} < c \exp(-\tfrac{1}{2}(c_T/c_b)^2),$$

where c is an absolute constant, while c_b is a function of H.

One proceeds in a similar manner to evaluate $p_{k,1}$:

$$y(x_1) - y(x_2) = \int_{x_2}^{x_1} (b_H(s) - b_H(k))\, dx + b_H(k)(x_1 - x_2).$$

If $\max[y(x_1) - y(x_2)] > c_T$ in $\Delta_k \times \Delta_k$ and $|b_H(k)| < c_T/2$, then

$$\max \int_{x_2}^{x_1} [b_H(s) - b_H(k)]\, ds > c_T/2.$$

Consequently,

$$p_{k,1} < P\left\{\max\left[\int_{x_2}^{x_1} (b_H(s) - b_H(k))\, ds, x_1, x_2 \in \Delta_k\right] > c_T/2\right\}$$
$$= P\left\{\max\left[\int_{x_1}^{x_2} b_H(s)\, ds, x_1, x_2 \in \Delta_0\right] > c_T/2\right\}.$$

Here again, we have used the relation $b_H(x) - b_H(k) \overset{\mathrm{d}}{=} b_H(x-k)$ with a fixed k. The use of the Fernique inequality[(4)] yields

$$p_{k,1} < c \exp(-\tfrac{1}{2}(c_T/2c_y)^2)$$

where c_y is a function of H. Combining the estimates of $p_{k,1}$ and $p_{k,2}$ and assuming $c_T = \max(c_b, 2c_y)\sqrt{2a \log T}$, one gets

$$p_k = p_{k,1} + p_{k,2} < cT^{-a}.$$

However, one then has $P(\mathscr{B}) < 2cT^{-a+1}$ and

$$P(A_T^+ < 1, |G_T| < T) \leqslant P(M_T < c_T) + O(T^{-a+1})$$
$$= P(M_{T'} < 1) + O(T^{-a+1}),$$

where $T' = cT(\log T)^{-\rho}$, $\rho = 0.5(1+H)^{-1}$. Hence $(A) \to (D)$.

Consider the second part of the theorem. Let $p_T(\Theta)$ be the probabilities that appear in the first part of the theorem, where Θ denotes the conditions A, B, C, or D. It has been shown above that, when $T \gg 1$,

$$p_T(A) < p_T(B)^{1-\varepsilon_T} \mathscr{L}_T, \qquad p_T(B) < p_T(C),$$
$$p_T(C) < p_{T'}(A) + O(T^{-a}), \qquad p_T(D) < p_{T'}(A) + O(T^{-a+1}),$$

where $a > 0$ is any fixed number, $\mathscr{L}_T$ is a slowly varying function of T, $\varepsilon_T = o(1)$, $T \to \infty$ and $T' = cT(\ln T)^{-\rho}$, $\rho = 0.5(1+H)^{-1}$. A trivial corollary of these is that all the $p_T(\Theta)$ have the asymptotics $\ln p_T(\Theta) = -\theta \ln T(1+o(1))$, provided the asymptotics is true for at least a single quantity of the type $\Theta = A, C$ or D.

Our proof has not relied significantly on the type of the interval Δ_T: $(-T, T)$ or $(0, T)$. For this reason our conclusion that the asymptotics of $p_T(\Theta)$ are identical also holds for $(0, T)$.

We conclude by noting that, if $\Delta_T = (0, T)$, then $p_T(B) = P(y(x) > 0, 1 < t < T)$. Consequently, if Z_T is the rightmost zero of $y(x)$ in $(0, T)$, then $P(Z_T < 1) = 2p_T(B)$.

3. THE GENERATION OF IFBM

We are going to use Monte Carlo techniques in order to evaluate the probabilities $p_T(\Theta)$ with $\Theta = A, C, D$ in Theorem 1 for the process $y(x) = \int_0^x b_H(s)\, ds$ in the following intervals of Δ_T: $(0, T)$ and $(-T, T)$. The probabilities in question are small, $p_T \to 0$ as $T \uparrow \infty$, hence the IFBM generation should be exact for a discrete sequence $\{x_k, k = 1,\dots, T\}$. Since $y(x)$ is a self-similar process, it is sufficient to use integer points $x_k = k$. In that case $\{y(k/T)\} \overset{\mathrm{d}}{=} \{T^{-(1+H)} y(k)\}$, while the probabilities $p_T(\Theta)$ can obviously be expressed in terms of the statistics $M = \max_{\Delta_1} y(x)$, $G = \arg\max_{\Delta_1} y(x)$, $A^+ = \int_{\Delta_1} \mathbf{1}_{y>0}\, dx$ of the process $\{y(x), x \in \Delta_1\}$, where $\Delta_1 = (0, 1)$ or $(-1, 1)$, as follows:

$$p_T(A) = F_M(T^{-(1+H)}); \qquad p_T(C) = F_{|G|}(T^{-1}); \qquad p_T(D) = \hat{F}_A(T^{-1})\, F_G(1-0) \tag{10}$$

where F_ξ is the distribution of ξ and $\hat{F}_A$ is the conditional distribution of A^+ given $|G| \neq 1$.

The Generation of $\{y(k), k=0,..., T\}$. The sequence $\{y(k), k=0,..., T\}$ is Gaussian and has stationary second increments, i.e., the sequence

$$\eta_k = y(k-1) - 2y(k) + y(k+1), \qquad k = 1,..., T-1,$$

has a Toeplitz correlation matrix $[\mu_{i-j}]$, where

$$\mu_k = c_q[|k-2|^q - 4\,|k-1|^q + 6\,|k|^q - 4\,|k+1|^q + |k+2|^q] \tag{11}$$

and $c_q = [2q(q-1)]^{-1}$, $q = 2H+2$.

The second differences $\{\eta_k\}$ combined with the initial conditions $y(0)=0$ and $y(1)$ are sufficient to uniquely reconstruct the sequence $\{y(k), k=0,..., T\}$. One can assign $y(1)$ by using the decomposion $y(1) = \hat{y}(1) + y^{\perp}(1)$ into the predictable part $\hat{y}(1) = E(y(1) \mid \eta_1,..., \eta_{T-1})$ of $y(1)$ and the part $y^{\perp}(1)$ that cannot be predicted from the data $\{\eta_i, i=1,..., T-1)\}$. In that case

$$y(1) = \sum_{k=1}^{T-1} z_k \eta_k + \sigma \varepsilon_0. \tag{12}$$

Here, $\mathbf{z} = (z_1,..., z_{T-1})'$ is the solution of the linear equation:

$$[\mu_{i-j}]_1^{T-1}\, \mathbf{z} = \mathbf{m}, \tag{13}$$

where the vector $\mathbf{m}$ has the components

$$m_k = Ey(1)\,\eta_k = \Delta[q\,|k|^{q-1} - |k|^q + |k-1|^q]\; c_q$$

and Δ is the difference operator of second order: $\Delta f(k) = f(k-1) - 2f(k) + f(k+1)$. The second term is $\sigma\varepsilon_0 = y^{\perp}(1)$, where ε_0 is the standard Gaussian random variable which is independent of $\{\eta_1,..., \eta_{T-1}\}$;

$$\sigma^2 = E[y^{\perp}(1)]^2 = q^{-1} - \sum_{1 \leqslant k < T} z_k m_k,$$

because $q^{-1} = E\,|y(1)|^2$ and $E\,|\hat{y}(1)|^2 = \sum_1^{T-1} z_k m_k$.

It thus appears that the exact generation of the sequence $\{y(k), k=0,..., T\}$ reduces to the generation of the stationary Gaussian sequence $\{\eta_k, k=1,..., T-1)\}$ with correlation function (11) and to the solution of the linear equation (13).

The Generation of $\{y(k), |k| < T/2\}$. For generating $y(x)$ in a bilateral interval, we note the following. Assume that $y(x)$ is IFBM in $(0, T)$, while $\tilde{y}(x)$ is IFBM in $(-T', T'')$, $T'+T''=T$ and $\tilde{y}(0)=\tilde{y}'(0)=0$. In that case

$$\{y(x)-y(T')-y'(T')(x-T'),\, x\in(0,T)\} \stackrel{d}{=} \{\tilde{y}(x-T'),\, x\in(0,T)\}.$$

The left-hand side provides a key to how one is to transform the sequence $\{y(k), k=0,..., T\}$ into an IFBM sequence that starts from the point $0<k_0<T$, i.e., $\tilde{y}(k_0)=\tilde{y}'(k_0)=0$. To do this one must also find the derivative $y'(k_0)$. In a similar way as above:

$$y'(k_0)=E\{y'(k_0)\mid \eta_1,..., \eta_{T-1}\}+E\{y'(k_0)\mid \varepsilon_0\}+y'^{\perp}(k_0)$$

where the first two terms correspond to the predictable part of $y'(k_0)$ based on the data $\{\eta_1,..., \eta_{T-1}, \varepsilon_0\}$, while the third term corresponds to the unpredictable part of $y'(k_0)$. The predictable part is

$$E\{y'(k_0)\mid \eta_1,..., \eta_{T-1}, \varepsilon_0\}=\sum_{k=1}^{T-1} z'_k\eta_k+a\varepsilon_0,$$

where $(z'_1,..., z'_{T-1})$ is the solution of (13) with the right-hand side $\mathbf{m}'=(m'_1,..., m'_{T-1})$. The components of $\mathbf{m}'$ are

$$m'_k=Ey'(k_0)\,\eta_k=\Delta[|k|^{q-1}+|k_0-k|^{q-1}\operatorname{sgn}(k_0-k)]\,qc_q.$$

One has $a=Ey'(k_0)\,\varepsilon_0$. From (12) one derives $Ey(1)\,y'(k_0)=\sigma a+\sum_{k=1}^{T-1} z_k m'_k$. Hence

$$\sigma a=c_q\cdot q[(q-1)\,k_0^{q-2}+1-k_0^{q-1}+(k_0-1)^{q-1}]-\sum_{k=1}^{T-1} z_k m'_k.$$

One has $y'^{\perp}(k_0)=\sigma'\varepsilon'$ where ε' is a standard Gaussian variable that is independent of $(\eta_1,..., \eta_{n-1}, \varepsilon)$. The variance of the unpredictable part σ'^2 can be found from the relation

$$k_0^{2H}=E[y'(k_0)]^2=\sum_{k=1}^{T-1} z'_k m'_k+a^2+\sigma'^2.$$

To sum up, the exact generation of $\{y(k), k=0,..., T;\, y(k_0)=y'(k_0)=0\}$ requires that an equation like (13) should be solved twice.

The Generation of $\{\eta_k\}$. Bardet *et al.*[2] provide a review of the methods which allow generation of Gaussian stationary sequences with a prescribed correlation function. We use the progressive Schur algorithm,[1] which is a Levinson–Durbin method. The Generalized Schur algorithm can be used in the framework of this method for fast solution of equations like (13) by the Gohberg–Semenkul formula.[1] The generation of $\{\eta_k, k=1,\dots, T-1\}$ by this method requires $O(T^2)$ operations. The computation is organized so as to minimize the amount of calculation needed for generating N IFBM samples; the computational complexity is a linear function of N and the storage capacity is of order $O(T^2)$.

The parameters N and T are equally important in the problem considered. The first parameter controlls the error ε_1 of the Monte Carlo method since $\varepsilon_1 = O(N^{-0.5})$, while the second one controlls the error ε_2 resulting from discretization. To assess the order of ε_2 we note the following: we are interested in the distributions of M, G, A^+ near zero where they are expected to behave like $x^\theta \mathcal{L}(x)$, $\mathcal{L}$ being a slowly varying function. Since the approximation to IFBM is discrete, these distributions contain a positive atom at zero of size $p_0(T) = P(y(k) \leqslant 0, k=0, \pm 1, \pm 2,\dots, k \in \Delta_T)$. The probability is doubled for the statistic Z (the rightmost zero of $y(t), t \in (0, T)$). Obviously $p_0(T) \geqslant P\{y(x) \leqslant 0, \forall x\colon |x| \geqslant 1, x \in \Delta_T)\}$ because $y(0)=0$. Therefore, if the bounds given by Theorem 1 are explicit, $p_0(T)$ and consequently ε_2 should not decrease faster than $T^{-\theta}\mathcal{L}'(T)$. The expected value is $\theta = 1-H$ for the interval $(-T, T)$ and $\theta \leqslant 1/4$ for $(0, T)$ (see below). The small value of $\theta(H)$ and run-time memory size (of order T^2) prevent from setting ε_2 equal to an arbitrary small value. For instance (see below), taking $T = 8194$ and $N = 50000$ we have $T^{-\theta} \approx 0.1$ and $N^{-0.5} \approx 0.0045$ if $\theta = 0.25$. Note however that these numbers tell not so much about the accuracy of exponent θ, since the convergency rate of the probability densities near zero (as $T \to \infty$) is unknown.

4. EVALUATION OF $\theta(H)$ AND RELATED RESULTS

The numerical analysis of log-asymtotics

$$\log p_T = -\theta(H) \log T(1+o(1)), \qquad T \to \infty$$

resulting from Theorem 1 and expression (10) can be reduced to an analysis of distribution function $F(x)$ as $x \to 0$. Here $F(x)$ denote the distribution function of either $M^{\frac{1}{1+H}}$, $|G|$, A^+, or Z (see Sections 1 and 3). These statistics correspond to the process $y(t)$, $t \in \Delta$, Δ being an interval of fixed length. By virtue of self-similarity of $y(t)$ we may set $\Delta = (-\frac{1}{2}, \frac{1}{2})$ for $\Delta_T = (-T, T)$ and $\Delta = (0, 1)$ for $\Delta_T = (0, T)$. Having $\log F(x) = \theta \log x(1+o(x))$ $x \to 0$ for any of these statistics we get much the same asymptotic for the other, moreover, the corresponding slopes coincide with $\theta(H)$—this results from the second part of Theorem 1.

We have analyzed all cases of these statistics, but we present here only the estimates for the maximum position $|G|$ since the conclusions and difficulties in the other cases are much the same. The distribution of $|G|$ is the key point for analysis of probabilities p_T, $\Delta_T = (-T, T)$ in a large class of self-similar processes. This is indicated by Theorem 2 which is also of some additional interest.

The distributions of $|G|$ are estimated for $H = 0.1 \div 0.9$ at increments of 0.1. The estimates result from $N = 50000$ samples of $y(t)$ with sample size $T = 8194$, the latter corresponding to the discretization $\delta = T^{-1} \approx 1.2 \cdot 10^{-4}$. The generation of $y(t)$ for different H uses identical white noise samples that is why the residuals of the corresponding estimates appeared to be correlated for different H.

We derive the estimates by Maximum Likelyhood method using the assumption $F(x) = Cx^{\theta}$ within an appropriate interval (x_-, x_+). Obviously $x_- > \delta$ since the discrete-time approximation of $F(x)$ keeps the atom $p_0(T) = O(T^{-\theta})$ at $x = 0$. Note that such a discontinuity causes a bend in the log-log plot of F for sufficiently small x.

The Bilateral IFBM Process

Figure 1 presents the distributions of $|G|$ for the bilateral IFBM process. The curves are well consistent with the log-log linear behaviour of $F(x)$ for small x. The estimates of the slope $\hat{\theta}(H)$ in intervals $\Delta_i = (10^{-3}, 10^{-2}) \cdot i$, $i = 1 \div 5$ are consistent with the hypothetical value $\theta(H) = 1 - H$ as well. The residual $|\hat{\theta} - \theta|$ in intervals Δ_i does not exceed 0.03. We have not taken into account interval $(\delta, 10\delta)$ where $\delta \sim 10^{-4}$, since all curves of $\log F(x)$ have biased slopes there due to discontinuity (of the discrete time distributions) at zero. Let us clarify the nature of the asymptotics

$$P(|G_T| < 1) = T^{-(1-H)} \mathscr{L}_T. \tag{14}$$

The function $y(x)$ is differentiable, hence the position of the global maximum, G_T, belongs to the zero set of $b_H(x)$ or to the end-points of $(-T, T)$. For this reason it should seem that the local time $l(x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^x \mathbf{1}_{|b_H(s)| < \varepsilon}\, ds$ is the natural time scale in our problem of the maximum of $y(x)$, i.e., it is more natural to study $\tilde{y}(l) = y(x(l))$ instead of $y(x)$, where $x(l)$ is the inverse function of $l(x)$ which is continuous on the right. The process was first treated by Vergassola *et al.*[20] and independently used by Isozaki and Watanabe[12] to prove the Sinai asymptotics for $H = 1/2$. It is a known fact[13] that $l(x)$ is a continuous self-similar process with parameter $h = 1 - H$. Consequently, $l(T) = O(T^{1-H})$, and (14) means,

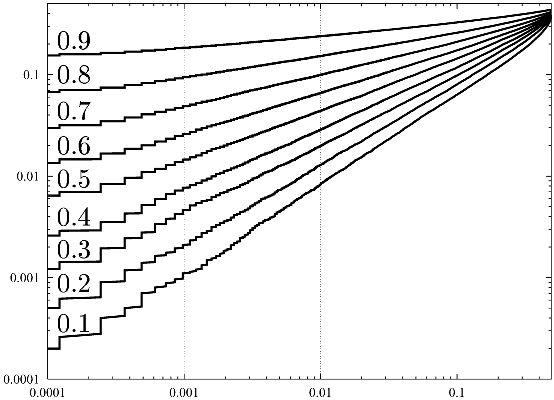

Fig. 1. Distributions of $|G|$ for the position of maximum G of the IFBM process in interval $(-\frac{1}{2}, \frac{1}{2})$. Various values of H are shown at the curves.

roughly speaking, that $P(|\tilde{G}_L| < 1) = L^{-1}\mathscr{L}_L$, $L \gg 1$ where $\tilde{G}_L$ is the location of the maximum of $\tilde{y}(l)$ in $(-L, L)$. Indeed, this relation holds for $H = 1/2$, because $\tilde{y}$ is a stable Levy process of index $\alpha = 1/3$. The theorem to follow shows that this relation also holds for general self-similar processes with stationary increments (SSSi).

Theorem 2. Let $\xi(t)$, $\xi(0) = 0$ be a cadlag (right-continuous with limits to the left) SSSi-process and $M = \sup\{\xi(s), s \in [0, 1]\} = \max(\xi(G), \xi(G-))$ where $0 \leqslant G \leqslant 1$ is the leftmost position of M. Then

1. G has a continuous probability density $\psi(t)$ in (0,1), and

$$\psi(t) \leqslant \psi(s) \max\left(\frac{s}{t}, \frac{1-s}{1-t}\right), \qquad \forall t, s \in (0, 1), \tag{15}$$

i.e., $\psi \equiv 0$ or $\psi > 0$.

2. if $\psi \not\equiv 0$, then the position G_T of the supremum of $\xi(t)$ in $(-Ta, T(1-a))$ satisfies the following relation:

$$P(|G_T| < 1) = \frac{\psi(a)}{2T}(1 + o(1)), \qquad T \to \infty. \tag{16}$$

The proof of Theorem 2 is nearly identical with that given in ref. 16 for the process $b_H(x)$, so it is relegated to the Appendix. Note that the asymptotics (16) appears as a simple consequence of the fact that the statistics of G has a nonzero distribution density. For stable Levy processes one has either

$\psi(t) = Ct^{\rho-1}(1-t)^{-\rho}$ (if $0 < \rho < 1$) or $\psi(t) = \delta(t-\rho)$ (if $\rho = 0$ or 1). Therefore bounds in (15) are accurate enough for the whole class of SSSi processes.

The Unilateral IFBM Process

The distribution of the position of the maximum G for $y(t)$, $0 < t < 1$ are presented by Fig. 2 (cases $H = 0.1 \div 0.4$—left, cases $H = 0.5 \div 0.9$—right). We have good confirmation of the log-log linear asymptotics near zero for those H for which the discontinuity $p_0(T)$ is small, i.e., $p_0(T) < 0.1$ as in Fig. 1. In addition one can assert that $\theta(\frac{1}{2}) > \theta(H)$ and $\theta(H) \to 0$ at the endpoints $H = 0, 1$. The latter is in agreement with the following representation of $y(t)$ in the degenerate cases: $y(t) = \xi t$ for $H = 0$ and $y(t) = \frac{1}{2}\xi t^2$ for $H = 1$, where ξ is a standard random Gaussian variable. As a result one has $P(G < x \mid H = 0, 1) = \frac{1}{2}$ for $x < 1$ and $\theta(0) = \theta(1) = 0$.

The estimates of θ in the interval of G: $10^{-3} - 10^{-2}$ are as follows:

H:	0.1	0.2	0.3	0.4	0.5
$\hat{\theta}$:	0.09	0.13	0.18	0.21	0.23
θ_0:	0.09	0.16	0.21	0.24	0.25

We also list the hypothetical values of θ: $\theta_0 = H(1-H)$ for comparison purposes in the above table. The exact result due to Sinai:[19] $\theta = 0.25$ for $H = 1/2$ shows that we can still take the error of $\hat{\theta}$ equal to 0.03. In that case the hypothetical estimates θ_0 do not contradict the empirical ones. All deviations $\hat{\theta} - \theta_0$ for listed $\theta_0(H)$ are negative that is in accordance with strong correlations of $\hat{\theta}$ for different H (see above).

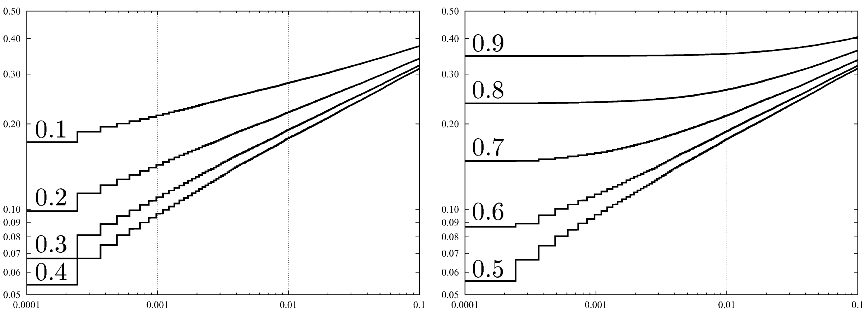

Fig. 2. Distribution of the position of maximum G for the IFBM process. Shown are the parts of curves related to the interval $(10^{-4}, 10^{-3})$ together with the corresponding values of H.

It would be natural to expect an analytical dependence of θ on H for the IFBM process. Consequently, the hypothesis $\theta_0(H)$ can also be extended to cover the case $H > 1/2$ which analysis is difficult. That extrapolation is exact for $H = 1$.

The rigorous result guarantees that $\theta(H)/(1+H)$ is decreasing for $H > 1/2$.

Theorem 3. (a) The distribution $F_M(x \mid H)$ for the maximum of

$$y_1(t) = \sqrt{2H+2} \int_0^t b_H(s)\, ds, \qquad 0 < t < 1 \tag{17}$$

increases with increasing H in the interval $(1/2, 1)$ for any fixed $x > 0$.

(b) If $F_M(x^{1+H} \mid H) = x^{\theta(H)} \mathscr{L}(x)$, $x \downarrow 0$ where $\mathscr{L}(x)$ is a slowly varying function, then $\theta(H)/(1+H)$ decreases with increasing H in $(1/2, 1)$.

Proof. The process (17) differs from $y(x)$ by the normalization $E\,|y_1(1)|^2 = 1$. Let $\xi_q(x) = y_1(x^\theta)$ where $q = 2H+2$, $\theta = q_0/q$, $q_0 = 2H_0 + 2$, and $H > H_0$.

Since IFBM is self-similar with parameter $h = H+1$, one has $E\,|\xi_q(x)|^2 = |x|^{q_0} = E\,|\xi_{q_0}(x)|^2$.

We show in the Appendix that

$$E\xi_q(x)\, \xi_q(y) \geqslant E\xi_{q_0}(x)\, \xi_{q_0}(y) \tag{18}$$

when $H > 1/2$. In that case the Slepian lemma[14] yields

$$P(\max_{[01]} \xi_q(x) < u) \geqslant P(\max_{[01]} \xi_{q_0}(x) < u).$$

However, $\max_{[01]} \xi_q(x) = \max_{[01]} \sqrt{2H+2} \int_0^x b_H(s)\, ds$ which proves the first part of the statement. The second part is an obvious corollary of the first.

5. CONCLUSION

We were testing the hypothesis that the maximum M of the integral of fractional Brownian motion of index H considered on finite segment I, $0 \in I$ has the distribution $F_M(x^{(1+H)}) = x^{\theta(H)} \mathscr{L}_H(x)$, $x \to 0$, where $\mathscr{L}_H$ is a slowly varying function. We have presented theoretical arguments and computational evidence to support and refine the hypothesis as follows: $\theta = 1 - H$ for $I = (-1, 1)$ and $\theta = \theta_+ = H(1-H)$ for $I = (0, 1)$. The first hypothesis seems to be certainty, while the second one is the simplest theoretical statement consistent with the data (see more in ref. 23).

Our computations reject the hypothesis $\theta_+ = (1-H)/2$ from ref. 20. Using Isozaki's approach, one can show that $\theta_+ = (1-H)/2$ holds for integral of Levy stable processes with Levy parameters $1 < \alpha \leqslant 2$, $|\beta| \leqslant 1$ and with the self-similarity parameter $H = \alpha^{-1} \in [\frac{1}{2}, 1)$. The case $H = 1/2$ is relevant to Brownian motion again. The estimate $\theta_+ = (1-H)/2$ is based on the fluctuation theory for random walks. New approaches are therefore required in the nonmarkovian case for analysis small maxima or long excursions. Theorems 1 and 2 in the present paper and the results of ref. 16 constitute a step in that direction.

We emphasize computational difficulties of estimating θ and demonstrate how these difficulties are related to the discretization of IFBM. One can not exclude the discretization effect considering the original problem (2) and using direct methods of estimation of the Hausdorff dimension of Lagrangian regular points (the case of θ for $I = (-1, 1)$). The difficulties one faces in this approach are discussed in refs. 9 and 18.

APPENDIX

Proof of Theorem 2. A distribution function (here F_G) is differentiable almost everywhere, because of monotonicity. Suppose this is true for the point x_0. Consider $x < x_0$, $\lambda = x_0/x > 1$. The self-similarity of $\xi(x)$ yields

$$P(G(0,1) \in dx) = P(G(0,\lambda) \in \lambda dx) \leqslant P(G(0,1) \in \lambda dx) = \psi(x_0)\frac{x_0}{x}\,dx. \tag{19}$$

Here, $G(a, b)$ is the leftmost position of the supremum of $\{\xi(x), x \in (a, b)\}$. Consequently, the distribution of $G(0, 1)$ is absolutely continuous in $(0, x_0)$. Points like x_0 are dense in $(0, 1)$. Consequently, $F_G(dx) = \psi(x)\,dx$, $x \in (0, 1)$.

In virtue of (19), $\psi(x)\,x$ is a nondecreasing function, i.e., the discontinuities in ψ are at most denumerable, while finite limits on the left and the right exist at the discontinuity points. The fact that the increments of $\xi(x)$ are stationary yields

$$\begin{aligned} P\{G(0,1) \in dx\} &= P\{G(-a, 1-a) \in dx-a\} \leqslant P\{G(0, 1-a) \in dx-a\} \\ &= P\left\{G(0,1) \in d\left(\frac{x-a}{1-a}\right)\right\} \end{aligned}$$

for any $0 < a < 1$. One has

$$\psi(x) \leqslant \psi\left(\frac{x-a}{1-a}\right)\frac{1}{1-a}$$

at continuous points of ψ. Multiply both parts by $(1-x)$:

$$(1-x)\,\psi(x) \leqslant \psi\left(\frac{x-a}{1-a}\right)\left(1-\frac{x-a}{1-a}\right) = \psi(y)(1-y), \qquad y = \frac{x-a}{1-a} < x.$$

Combining both inequalities, one gets

$$\psi(x) \leqslant \psi(y)\max\left(\frac{y}{x}, \frac{1-y}{1-x}\right) \tag{20}$$

at all continuous points x and y. In particular, $\psi(x+0) \leqslant \psi(x-0) \leqslant \psi(x+0)$, i.e., ψ is continuous in $(0, 1)$. If $\psi(x_0) = 0$, $x_0 \in (0, 1)$, then one has $\psi(x) = 0$, $x \in (0, 1)$ from (20). Consequently, the following alternative holds: either $\psi \equiv 0$ or $\psi > 0$ in $(0, 1)$. The second part of Theorem 2 is an immediate corollary of the first part and the self-similarity of $\xi(x)$, see ref. 16.

The Proof of (18) in Theorem 3. Let $\xi_q(t) = \sqrt{q}\int_0^\tau b_H(s)\,ds$, $\tau = t^\theta$, $\theta = q_0/q < 1$, $q = 2H+2$. The correlation function $\beta_q(t, s)$ of $\xi_q(t)$, can be written as

$$2t^{-q_0}\beta_q(t, s) = \frac{q_0}{q_0-\theta}(\rho^\theta + \rho^{q_0-\theta}) + [(1-\rho^\theta)^{q_0/\theta} - (1+\rho^{q_0})]\frac{\theta}{q_0-\theta} \tag{21}$$

where $\rho = s/t$. Because β_q is symmetric in t, s, we put $\rho \leqslant 1$. We will show that $\beta_q(t, s) > \beta_{q_0}(t, s)$, if $q > q_0 > 3$ or, which amounts to the same thing, $H > H_0 > 1/2$.

We have in virtue of (21):

$$2t^{-q_0}[\beta_q(t, s) - \beta_{q_0}(t, s)] = (q-q_0)(1-\rho^\theta)(q_0-1)^{-1}R_1 + q_0(q_0-1)^{-1}R_2.$$

Here $R_1 = (1-y^{q-1}-\bar{y}^{q-1})(q-1)^{-1} - (\bar{y}^{q_0-1}-\bar{y}^{q-1})(q-q_0)^{-1}$, $y = \rho^\theta$, $\bar{y} = 1-y$ and

$$R_2 = \int_\theta^1 [\rho^\alpha - \rho^{q_0-\alpha} - (1-\rho^\alpha)^{q_0-1}\rho^\alpha]\,d\alpha \ln 1/\rho.$$

We now are going to show that $R_1 \geqslant 0$, and $R_2 \geqslant 0$, if $H \geqslant 1/2$.

Consider R_2. Put $\rho^\alpha = u$. Since $0<\theta<\alpha<1$ and $0<\rho<1$, it follows that $0<u<1$. The integrand in R_2 becomes

$$u[1-u^{q_0/\alpha-2}-(1-u)^{q_0-1}] \geqslant u[1-u^{q_0-2}-(1-u)^{q_0-2}]$$
$$\geqslant u[1-\max(1, 2^{3-q_0})] \geqslant 0.$$

The last estimate is true, because $3-q_0 = 1-2H \leqslant 0$ when $H>1/2$. Consequently, $R_2 \geqslant 0$.

Consider R_1. The function $R_1(y)$ is positive around 0 and 1:

$$R_1 = \begin{cases} y^2(q_0-1)/2+O(y^{q-1}), & y\to 0 \\ \bar{y}+O(\bar{y}^{q_0-1}), & \bar{y}\to 0. \end{cases}$$

Consequently, $R_1 \geqslant 0$, if the function has a single local extremum in $(0, 1)$. Let $z=(1-y)^{-1} \in (1, \infty)$. Then

$$-z^{q_0-2}\frac{d}{dy}R_1 = [(z-1)^{q-2}-1]-[z^{q-q_0}(q_0-1)-q-1](q-q_0)^{-1} := f(z).$$

We now show that $f(z)$ has a single root in $(1, \infty)$. The function

$$f(z) = \begin{cases} -(z-1)(q_0-1)+O((z-1)^{2H}), & z\to 1 \\ (z-1)^{q-2}\,(1+o(1)), & z\to\infty \end{cases} \tag{22}$$

changes sign in $(1, \infty)$. The equation $f'(z)=0$ or

$$(q-2)(z-1)^{q-3} = (q_0-1)\,z^{q-q_0-1} \tag{23}$$

determines the local extremums of $f(z)$ in $(1, \infty)$. Two strictly monotone functions occur in (23): on the left is a function that increases from 0 to ∞, because $q-3=2H-1>0$, while the nonnegative function on the right decreases toward zero at infinity, because $q-q_0-1 = 2(H-H_0)-1 < 2\cdot 1/2-1<0$. In that case, however, (23) has the single root $1<z^*<\infty$. In virtue of (22) z^* is the point of minimum, where $f(z^*)<0$. The function $f(z)$ is strictly increasing from $f(z^*)<0$ to ∞ in (z^*, ∞), so that the equation $f(z)=0$ has a single root, as was to be proved.

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